Linear algebra - Practice problems for final

1. Diagonalize the matrix $\begin{bmatrix} 3 & 0 & 0 \\ -3 & 4 & 9 \\ 0 & 0 & 3 \end{bmatrix}$.

Solution.

To find the eigenvalues, compute

$$\det \begin{bmatrix} 3-\lambda & 0 & 0 \\ -3 & 4-\lambda & 9 \\ 0 & 0 & 3-\lambda \end{bmatrix} = (3-\lambda)(4-\lambda)(3-\lambda).$$

So the eigenvalues are $\lambda = 3$ and $\lambda = 4$.

We can find two linearly independent eigenvectors $\begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$ corresponding to the eigenvalue 3, and one eigenvector $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ with eigenvalue 4. The diagonalized form of the matrix is

$$\begin{bmatrix} 3 & 0 & 0 \\ -3 & 4 & 9 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & -3 \\ -3 & 1 & 9 \end{bmatrix}.$$

Note that if you chose different eigenvectors, your matrices will be different. The middle matrix should have entries $3, 3, 4$ in some order, and you should multiply out the product to make sure you have the right answer.

2. Find a formula for $\begin{bmatrix} 1 & -6 \\ 2 & -6 \end{bmatrix}^k$ by diagonalizing the matrix.

Solution. The eigenvalues are $-3, -2$, and the diagonalized form of the matrix is

$$\begin{bmatrix} 1 & -6 \\ 2 & -6 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} -3 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 2 & -3 \end{bmatrix}.$$

It follows that

$$\begin{bmatrix} 1 & -6 \\ 2 & -6 \end{bmatrix}^k = \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} (-3)^k & 0 \\ 0 & (-2)^k \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 2 & -3 \end{bmatrix}.$$

3. Let $B = \{\mathbf{b}_1, \mathbf{b}_2\}$ and $C = \{\mathbf{c}_1, \mathbf{c}_2\}$ be two bases for $\mathbf{R}^2$ such that $\mathbf{b}_1 = 6\mathbf{c}_1 - 2\mathbf{c}_2$ and $\mathbf{b}_2 = 9\mathbf{c}_1 - 4\mathbf{c}_2$.

(a) Find the change of coordinates matrix from B to C.

(b) If the vector $\mathbf{v}$ has coordinate vector $\mathbf{v}_B = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$, find $\mathbf{v}_C$.

Solution.

(a) The change of coordinate matrix from B to C is $\begin{bmatrix} 6 & 9 \\ -2 & -4 \end{bmatrix}$. Note that for instance the fact that

$$\begin{bmatrix} 6 & 9 \\ -2 & -4 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ -2 \end{bmatrix}$$

corresponds to the statement that the coordinate vector of $\mathbf{b}_1$ relative to the basis B is $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, whereas relative to the basis C it is $\begin{bmatrix} 6 \\ -2 \end{bmatrix}$.

(b) $\mathbf{v}_C = \begin{bmatrix} 6 & 9 \\ -2 & -4 \end{bmatrix} \begin{bmatrix} -3 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \end{bmatrix}$.

4. Find the projection of $\mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$ onto the subspace $W = \mathrm{sp}\left(\begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right)$ of $\mathbf{R}^4$.

Solution. First find the orthogonal complement of W. This is the nullspace of the matrix $\begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$. A basis for this nullspace (found by row reduction) is

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 1 \end{bmatrix} \right\}.$$

Now we need to find the coordinate vector of $\mathbf{b}$ relative to the basis

$$\left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 1 \end{bmatrix} \right\}$$

Again, by row reduction, we find the coordinate vector $\begin{bmatrix} -2/3 \\ 1/3 \\ 1 \\ 2/3 \end{bmatrix}$. To find the projection of $\mathbf{b}$ onto W we only take the part of the coordinate vector that corresponds to basis elements in W. We get

$$\mathbf{b}_W = \frac{-2}{3} \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} + \frac{1}{3} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -2/3 \\ -1/3 \\ 1/3 \end{bmatrix}.$$

5. Find the projection matrix onto the subspace $W = \mathrm{sp}\left(\begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right)$ of $\mathbf{R}^4$. Use this to compute the projection of the vector $\begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix}$ onto W.

Solution. Let $A = \begin{bmatrix} 1 & -1 \\ 2 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$. The projection matrix is $A(A^TA)^{-1}A^T$. Computing this we get

$$A(A^TA)^{-1}A^T = \frac{1}{17}\begin{bmatrix} 14 & 1 & 5 & -4 \\ 1 & 11 & 4 & 7 \\ 5 & 4 & 3 & 1 \\ -4 & 7 & 1 & 6 \end{bmatrix}.$$

The projection of $\begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix}$ onto W is then

$$\begin{bmatrix} 14 & 1 & 5 & -4 \\ 1 & 11 & 4 & 7 \\ 5 & 4 & 3 & 1 \\ -4 & 7 & 1 & 6 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix} = \frac{1}{17}\begin{bmatrix} 21 \\ 27 \\ 16 \\ 11 \end{bmatrix}.$$

6. Find the best fit line $y = mx + c$ through the data points $(0,0), (1,1), (2,3)$.

Solution. Finding a line $y = mx + c$ through those points amounts to solving the system

$$\begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} m \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}.$$

This system is inconsistent (since there is no line through those points), and to find the best fit line, we need to project the vector $\mathbf{b} = \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}$ onto the column space $W = \mathrm{sp}\left(\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right)$ of the matrix. The projection is

$$\mathbf{b}_W = \frac{3}{2}\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + \frac{-1}{6}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1/6 \\ 8/6 \\ 17/6 \end{bmatrix}.$$

The best fit line is given by the solution of

$$\begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} m \\ c \end{bmatrix} = \begin{bmatrix} -1/6 \\ 8/6 \\ 17/6 \end{bmatrix},$$

which is $m = 3/2$, $c = -1/6$.